Mu’tah University
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ON THE RADIUS OF TRANSNORMAL SPHERICAL PARTIAL TUBES

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DEDICATION

To my parents (Mercy be upon them).

Ahmad Al-Hoety

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First of all and always most thanks to ALLAH for creating me and guiding me in all aspect of my life.

I would like to express my sincere gratitude to my supervisor, Dr. Kamal Al-Banawi for his supervision, encouragement, continuous stimulating out all the stages of this work.

I pray to ALLAH to keep him for his wide support, because without his helping, my work could have been infinitely more difficult.

Finally, I am deeply thankful to my family specially my brother , sisters and wife for their support and help during my study years.

Ahmad Al-Hoety

LIST OF CONTENTS

TABEL OF FIGURE

ABSTRACT
ON THE RADIUS OF TRANSNORMAL SPHERICAL PARTIAL TUBES

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Transnormal spherical partial tubes is a special type of transnormal manifolds which have certain properties regarding their generating frames in Euclidean spaces .

In this thesis radii of transnormal spherical partial tubes are studied using concepts of differential geometry. The progress here is in the estimation of the radus of a transnormal embedding of a sphere bundle over a particular base with image a transnormal partial tube.

الملخص

حول نصف قطر الأنابيب الجزئية الكروية ذوات النقل العمودي

أحمد سليمان الحويطي

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الأنابيب الجزئية الكروية ذوات النقل العمودي هي نوع خاص من متعددات الطيات التي تمتلك خواصاً معينة تتعلق بإطارات التوليد في الفضاءات الإقليدية.

في هذه الأطروحة تُدرس انصاف اقطار الأنابيب الجزئية الكروية ذوات النقـل العمودي باستخدام مفاهيم الهندسة التفاضلية. إن الإنجاز هنا هو في حساب نصف قطر انغماس ذي نقل عمودي لتشعب كروي على قاعدة معينة لها صورة أنبوب جزئي.

Chapter One
Introduction

1.1 Preface

This thesis is concerned with the radius of a transnormal spherical partial tube , which is a research point that is a part of a general relatively new idea called transnormality. The concept of transnormality is due to S.Robertson [14,15,16].

In this thesis we take V as a smooth (C^∞) compact connected m-manifold without boundary, smoothly embedded in R^n for some $n > m$. We write $N(p)$ to mean the $(n\text{-}m)$-plane which is normal to V at p. The manifold V is transnormal in R^n whenever $q \in N(p)$ implies $N(q) = N(p)$ for all $p,q \in V$. The set $V \cap N(p)$ is called the generating frame of V at p.

In this thesis, we study the radius of a transnormal spherical partial tube. Mainly, our work is built on the work of S.Roberson [14,15,16], S.Carter and K.Al- Banawi [1,2,3]. The most recent work on transnormality has been done by Al-Banawi in the form of a study of focal points of transnormal tori in R^4 [5].

Studying the radius of a transnormal spherical partial tube is another step on the route of tubular neighborhoods which was introduced by S.Carter and A.West in [9]. The idea will be explained later, even though, it is about embedding M as a smooth connected m-manifold without boundary by a smooth function $f : M \to R^n$ into the Euclidean space R^n , $n = m + k$. Then we consider the normal bundle of M in which a smooth subbundle with some type fibre is embedded . If the end point map is restricted to that fibre, then we get a partial tube around f. The manifold $f(M)$ is usually called the base of the partial tube. A partial tube is spherical if the fibre is a sphere.

1.2 Aims and Objectives

The main aim of this thesis is to deduce new results concerning the radius of a transnormal spherical partial tube, which is the essence of Chapter 4. The research aims to build a strategy of calculating the radius of a sphere bundle over a particular base with image a transnormal partial tube. Then the generating frame of such a transnormal spherical partial tube will be studied by its radius and width. Examples are built to support the aim and the results.

1.3 Summary of Thesis

This thesis consists of four chapters including this chapter, the introduction.

Chapter **2** deals with the main concepts needed to understand the preliminaries of the work. The main content of Chapter **2** as follows.

A brief description of curves in R^2 is introduced [13]. Convex curves are defined there as in [10]. Also curves of constant width in R^2 are studied using the ideas of Mellish[12]. Chapter **2** summarizes the literature review of a curve of constant width in R^2 [4,6] and discusses a particular example [11]. A brief description of surfaces in R^3 is introduced [13]. Also surfaces of constant width in R^3 are studied [12]. Chapter **2** also introduces a brief idea regarding the literature review of a surface of constant width in R^3.

Chapter **3** introduces a brief description of differentiable manifolds [7]. Then the concept of a transnormal manifold is introduced as a base for the research in this thesis as in [16]. Usually, a transnormal manifold is described by its generating frame. Chapter **3** summarizes the literature review of transnormal manifolds [8,14,15,16], and explains many properties concerning transnormal mamifolds as well the generating frames of transnormal manifolds.

Chapter **4** contains the main results and contributions within this thesis. The ideas are mainly about the radius of a transnormal spherical partial tube. Transnormal embeddings of a sphere bundle over a particular base with image a partial tube are discussed. The generating frame of such a partial tube are studied and examples are built regarding a 4-transnormal embedding of a torus around a 2-transnormal curve (of constant width) in R^4.

Chapter Two
Preliminaries

In this chapter we introduce some concepts regarding the differential geometry of curves, surfaces and manifolds.

2.1 Curves in R^2

Definition 2.1.1. [13] A *curve* C in R^2 is a differentiable function $\vec{f}: \mathrm{I} \to R^2$ where I is an (open) interval.

Thus, if t is a parameter of $\vec{f}$, $t \in \mathrm{I}$, then we write $\vec{f}$ as

$$\vec{f}(t) = (f_1(t), f_2(t))$$

where f_1 and f_2 are differentiable real valued functions defined on I. We say that $\vec{f}$ is a parametrization of C.

Definition 2.1.2. [13] A curve C in R^2 with parametrization $\vec{f}: \mathrm{I} \to R^2$ is *regular* provided that $\vec{f}'(t) \neq \vec{0}$ for all $t \in \mathrm{I}$. If $\vec{f}'(t_1) = \vec{0}$ for some $t_1 \in \mathrm{I}$, then $\vec{f}(t_1)$ is a *singular* point of C.

Now we take $\mathrm{I} = [a, b]$ and define the *arclength* along C by

$$s(t) = \int_a^t \left\| \vec{f}'(u) \right\| du \qquad (2.1)$$

where $\left\| \vec{f}'(t) \right\|$ is the norm of $\vec{f}'(t)$. If C is a regular curve, then $\left\| \vec{f}'(t) \right\| > 0$ for all $t \in \mathrm{I}$ and so s is a strictly increasing function of t which has an inverse. That is, Equation (2.1) can be solved for s and then C can be reparametrized by arclength. Reparametrizing a regular curve by acrlength simplifies calculations regarding geometry of curves.

Theorem 2.1.3. [13] Let $\vec{f}$ be an arbitrary regular curve. Then the curvature of $\vec{f}$ is given by

$$\kappa(t) = \frac{\left\| \vec{f}'(t) \times \vec{f}''(t) \right\|}{\left\| \vec{f}'(t) \right\|^3}.$$

Definition 2.1.4. [13] The *focal curve* of a regular curve $\vec{f}$ is the curve $\vec{g} = \vec{f} + \frac{1}{\kappa}\vec{v}$ where $\vec{v}$ is the unit normal of $\vec{f}$. Similarly, the focal point of $\vec{f}$ with base $\vec{f}(t)$ is the point $\vec{g}(t) = \vec{f}(t) + \frac{1}{\kappa(t)}\vec{v}(t)$.

Now let $\Lambda_{\vec{p}}(t) = \left\|\vec{f}(t) - \vec{p}\right\|^2$ be the distance function whose domain is the curve C with parametrization $\vec{f}$.

Theorem 2.1.5. [1] The point $\vec{p}$ is a focal point of C with base $\vec{f}(t)$ iff

$$\Lambda'_{\vec{p}}(t) = 0 \text{ and } \Lambda''_{\vec{p}}(t) = 0.$$

Example 2.1.6. Let $\vec{f}(t) = (t, t^2)$. Then

$$\vec{f}'(t) = (1, 2t) \qquad \text{and} \qquad \vec{f}''(t) = (0, 2).$$

So $\qquad \left\|\vec{f}'(t)\right\| = \sqrt{1+4t^2} \qquad$ and $\qquad \left\|\vec{f}'(t) \times \vec{f}''(t)\right\| = 2.$

Thus,

$$\kappa(t) = \frac{2}{(\sqrt{1+4t^2})^3}.$$

Since $\vec{f}'$ represents the tangent, a choice for the unit normal of $\vec{f}$ is

$$\vec{v}(t) = \frac{(-2t, 1)}{\sqrt{1+4t^2}}.$$

Thus,

$$\vec{g}(t) = (t, t^2) + \frac{(\sqrt{1+4t^2})^3}{2} \frac{(-2t, 1)}{\sqrt{1+4t^2)}} = (-4t^3, \frac{1}{2} + 3t^2).$$

Now $\qquad \Lambda_{\vec{p}}(t) = (t-a)^2 + (t^2 - b)^2$ with $\vec{p} = (a, b)$. So

$$\Lambda'_{\vec{p}}(t) = (2-4b)t + 4t^3 - 2a$$

and

$$\Lambda''_{\vec{p}}(t) = 2 - 4b + 12t^2.$$

Equating the last two equations to zero, we get

$$b = 1/2 + 3t^2 \qquad \text{and} \qquad a = -4t^3.$$

Example 2.1.7. Consider the curve defined by :

$$\vec{f}(t) = (9\cos t + \cos t \cos 3t + 3\sin t \sin 3t, 9\sin t + \sin t \cos 3t - 3\cos t \sin 3t)$$

Then

$$\vec{f}'(t) = (9 - 8\cos 3t)(-\sin t, \cos t)$$

and

$$\vec{f}''(t) = (9 - 8\cos 3t)(-\cos t, -\sin t) + 24\sin 3t(-\sin t, \cos t).$$

So

$$\left\|\vec{f}'(t)\right\| = 9 - 8\cos 3t \qquad \text{and} \qquad \left\|\vec{f}'(t) \times \vec{f}''(t)\right\| = (9 - 8\cos 3t)^2.$$

Thus,

$$\kappa(t) = \frac{1}{9 - 8\cos 3t}.$$

A choice for the unit normal of $\vec{f}$ is $\vec{v}(t) = (-\cos t, -\sin t)$.
Thus,
$g(t) = (9\cos t\cos 3t + 3\sin t\sin 3t, 9\sin t\cos 3t - 3\cos t\sin 3t)$.
Let $\bar{p} = (a,b)$. Then
$\Lambda_{\bar{p}}(t) = (9\cos t + \cos t\cos 3t + 3\sin t\sin 3t - a)^2 + (9\sin t + \sin t\cos 3t - 3\cos t\sin 3t - b)^2$

So $\Lambda'_{\bar{p}}(t) = 2(9 - 8\cos t)(-3\sin 3t + a\sin t - b\cos t)$,

$\Lambda''_{\bar{p}}(t) = 2(9 - 8\cos 3t)(-9\cos 3t + a\cos t + b\sin t) + 48\sin 3t(-3\sin 3t + a\sin t - b\cos t)$.

Equating the last two equations to zero, we get

$$-3\sin 3t + a\sin t - b\cos t = 0$$

and

$$-9\cos 3t + a\cos t + b\sin t = 0.$$

Solving the last two equations together, we get
$a = 9\cos t\cos 3t + 3\sin t\sin 3t$ and $b = 9\sin t\cos 3t - 3\cos t\sin 3t$.

2.2 Curves of Constant Width in R^2

By a smooth function we mean a differentiable function, usually written as a C^∞ function. We start with a smooth closed curve C in the Euclidean plane R^2 parametrized by the arclength s. That is, C is defined by $\vec{f}(s) = (f_1(s), f_2(s))$ where the parameter s belongs to $[a,b]$ such that $\vec{f}(a) = \vec{f}(b)$. So our research point takes C as a simple closed curve, i.e. C doesn't join up except at the end points. Then we restrict our point of research to be about convex simple closed curves in R^2. We define convexity in two different but equivalent manners.

Definition 2.2.1. [10] A smooth simple closed curve C in R^2 is *convex* (or an *oval*) if C lies entirely at one side of it's tangent at any point chosen on it.

Definition 2.2.2. [10] A smooth simple closed curve in R^2 is *convex* (or an *oval*) if the curvature of C is strictly positive at each point on C.

Assume that $\vec{f}$ parametrizes C in an anticlockwise direction so that the bounded component of $R^2 - C$ is on the left. Let $\bar{\tau}$ and $\bar{v}$ be respectively the unit tangent and the unit normal fields acting on C. Thus, at each point $\vec{f}(s)$ on the oval there is a unique unit tangent vector $\bar{\tau}(s) = \frac{d\vec{f}}{ds}(s)$ in the direction of the oval and a unique inward-pointing unit normal vector $\bar{\upsilon}(s)$. Also for each $s \in R$, there is a unique $s^* \in R$ such that $\bar{\tau}(s) + \bar{\tau}(s^*) = 0$, and then $\bar{\upsilon}(s) + \bar{\upsilon}(s^*) = 0$. It is natural to say that $\bar{p}^* = \vec{f}(s^*)$ is *opposite* to $\bar{p} = \vec{f}(s)$.

Let T and N denote the lines in R^2 tangent and normal to C at $\bar{p}$, and let T^* and N^* denote the corresponding lines at $\bar{p}^*$. Let $a: R \to R$ be the function that assigns to each $s \in R$ the perpendicular displacement between T and T^* and $b: R \to R$ the function that assigns to each $s \in R$ the perpendicular displacement between N and N^* (Figure 2.1).

The *width* of C at $\bar{p}$, or in the direction $\bar{\tau}(s)$, is defined as the number $a(s)$. In general, the width of an oval varies with $\bar{p}$.

Definition 2.2.3. [12] An oval is a *curve of constant width* if the perpendicular distance between tangent lines at opposite points $\bar{p}$ and $\bar{p}^*$ is independent of $\bar{p}$, i.e. a is constant.

The most obvious example of a curve of constant width a is a circle of diameter a.

Theorem 2.2.4. [12] A curve C in R^2 is of constant width a iff the affine normal lines at opposite points coincide .

Theorem 2.2.5. [4] A curve C is of constant width a iff opposite points have the same focal point.

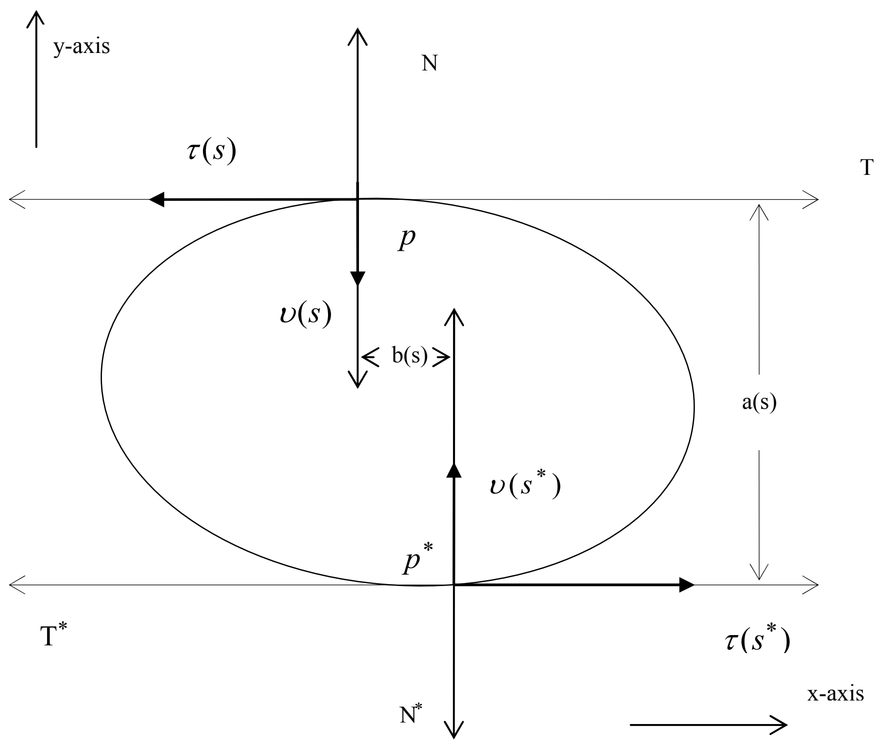

Figure 2.1:
An oval in R^2 with opposite points $\bar{p}$ and $\bar{p}^*$.

The next example was suggested by Fillmore in [11]. It is a curve of constant width in R^2 other than the circle.

Example 2.2.6. Consider the curve $\vec{f}: R \to R^2$ defined by

$\vec{f}(\theta) = (9\cos\theta + \cos\theta\cos 3\theta + 3\sin\theta\sin 3\theta, 9\sin\theta + \sin\theta\cos 3\theta - 3\cos\theta\sin 3\theta)$

where θ is taken mod 2π.

Since $$\frac{d\vec{f}}{d\theta} = (9 - 8\cos 3\theta)(-\sin\theta, \cos\theta),$$

clearly, $$\vec{\tau}(\theta) = (-\sin\theta, \cos\theta).$$

Since $$\vec{\tau}(\theta + \pi) = (\sin\theta, -\cos\theta) = -\vec{\tau}(\theta),$$

we have $$\forall\theta \in R,\ \vec{\tau}(\theta) + \vec{\tau}(\theta + \pi) = 0,$$

and so the points $\vec{p} = \vec{f}(\theta)$ and $\vec{p}^* = \vec{f}(\theta + \pi)$ are opposite points.

If $d(\vec{p}, \vec{p}^*)$ denotes the Euclidean distance between $\vec{p}$ and $\vec{p}^*$, then $\forall\theta \in R$, $d(\vec{p}, \vec{p}^*) = 18$, and so the curve is of constant width 18.

2.3 Surfaces in R^3

Let U be an open set in R^2 where R^2 has the standard topology.

We define the map $\bar{F}: U \to R^3$ by assuming that $(u, v) \in U$ and

$$\bar{F}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

Defintion 2.3.1. [13] The map $\bar{F}: U \to R^3$ is a regular parametrization if $\bar{F}_u$ and $\bar{F}_v$ both exist on U and $\bar{F}_u \times \bar{F}_v \neq \bar{0}$ for all $(u, v) \in U$.

Example 2.3.2. The map $\bar{F}: U \to R^3$ with $U = \{(u, v) \in R^2 : u^2 + v^2 \prec 1\}$ and

$$\bar{F}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

is a regular parametrization. Here U is open in R^2 and

$$\bar{F}_u = (1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}})$$

$$\bar{F}_v = (0, 1, \frac{-v}{\sqrt{\sqrt{1 - u^2 - v^2}}})$$

Both exist with $\bar{F}_u \times \bar{F}_v = (\frac{u}{\sqrt{1 - u^2 - v^2}}, \frac{v}{\sqrt{1 - u^2 - v^2}}, 1) \neq \bar{0}$ on U.

Example 2.3.3. The map $\bar{F}: U \to R^3$ with $U = \{(u, v) \in R^2 : u \succ 0, v \succ 0\}$ and

$$\bar{F}(u, v) = (u^2, v^2, uv)$$

is a regular parametrization. Here U is open in R^2 and

$$\bar{F}_u = (2u, 0, v),$$

$$\bar{F}_v = (0, 2v, u)$$

Both exist with $\bar{F}_u \times \bar{F}_v = (-2v^2, -2u^2, 4uv) \neq \bar{0}$ on U.

Definition 2.3.4.[13] The map $\bar{F}: U \to R^3$ is a coordinate patch if U is open in R^2, $\bar{F}$ is a regular parametrization and $\bar{F}$ is a homeomorphism onto its image.

Example 2.3.5. The map in Example 2.3.2 is a coordinate patch .It is one to one and continuous on U. Also the map $\bar{F}^{-1}: \bar{F}(U) \to U$ defined by

$$\bar{F}^{-1}(x, y, z) = (x, y)$$

with $z = \sqrt{1 - x^2 - y^2}$ is continuous on U.

Example 2.3.5. The map in Example 2.3.2 is a coordinate patch .It is one to one and continuous on U. Also the map $\bar{F}^{-1}: \bar{F}(U) \to U$ defined by

$$\bar{F}^{-1}(x, y, z) = (x, y)$$

with $z = \sqrt{1 - x^2 - y^2}$ is continuous on U.

Example 2.3.6. The map in Example 2.3.3 is a coordinate patch.It is one to one and continuous on U. Also the map $\bar{F}^{-1}: \bar{F}(U) \to U$ defined by

$$\bar{F}^{-1}(x, y, z) = (\sqrt{x}, \sqrt{y})$$

with $z = \sqrt{xy}$, $x \succ 0, y \succ 0$ is continuous on U.

Definition 2.3.7. [13] A surface S in R^3 is a subset of R^3 such that every point of S is covered by a coordinate patch. The set of coordinate patches covering S is called an *atlas*.

Example 2.3.8. The sphere S^2 is a surface. It is covered by six patches similar to the one in Example 2.3.2

Theorem 2.3.9.[13] Let h be differentiable real valued function defined on R^3 and $c \in R$. Let S be the nonempty subset of R^3 defined by $h(x, y, z) = c$.

If $dh(p) \neq \bar{0}$ for all $p \in S$, then S is a surface.

Example 2.3.10. The set $S = \{(x, y, z) \in R^3 : z = x^2 + y^2\}$ is a surface. For, let

$$h(x, y, z) = z - x^2 - y^2.$$

Then

$$dh = dz - 2xdx - 2ydy \neq \bar{0}$$

for all $(x, y, z) \in S$.

Example 2.3.11. The set $S = \{(x, y, z) \in R^3 : z^2 = x^2 + y^2\}$ is not a surface. For, let

$$h(x, y, z) = z^2 - x^2 - y^2.$$

Then

$$dh = 2zdz - 2xdx - 2ydy.$$

Now $dh = \bar{0}$ if $(x, y, z) = \bar{0} \in S$.

We start with $\bar{F}$ as a coordinate patch on S. Then we assume that $\bar{f}$ is a regular curve whose image lies on S. If t is the parameter of $\bar{f}$ and (u, v) is a local coordinate system of $\bar{F}$, then $f(t) = \bar{F}(u(t), v(t))$ for some smooth real valued functions u and v.

Now

$$\bar{f}' = \bar{F}_u u' + \bar{F}_v v'.$$

Thus,

$$\bar{f}' \cdot \bar{f}' = \bar{F}_u \cdot \bar{F}(u')^2 + 2\bar{F}_u \cdot \bar{F}_v (u'v') + \bar{F}_v \cdot \bar{F}_v (v')^2.$$

That is

$$\bar{f}' \cdot \bar{f}' = [u' \quad v'] \begin{bmatrix} \bar{F}_u \cdot \bar{F}_u & \bar{F}_u \cdot \bar{F}_v \\ \bar{F}_u \cdot \bar{F}_v & \bar{F}_v \cdot \bar{F}_v \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix},$$

which is equivalent to

$$\bar{f}' \cdot \bar{f}' = \bar{W}^T A \bar{W} ,$$

where $\bar{W} = \begin{bmatrix} u' \\ v' \end{bmatrix}$ and A is the well known first fundamental matrix.

A choice for a unit on S is $\hat{n} = \dfrac{\bar{F}_u \times \bar{F}_v}{\|\bar{F}_u \times \bar{F}_v\|}$.

Now

$$\bar{f}'' = \bar{F}_{uu}(u')^2 + 2\bar{F}_{uv}(u'v') + \bar{F}_u u'' + \bar{F}_{vv}(v')^2 + \bar{F}_v v''.$$

Thus

$$\bar{f}'' = (\bar{F}_{uu} \cdot \hat{n})(u')^2 + (2\bar{F}_{uv} \cdot \hat{n})(u'v') + (\bar{F}_{vv} \cdot \hat{n})(v')^2.$$

That is

$$\bar{f}'' \cdot \hat{n} = [u' \quad v'] \begin{bmatrix} \bar{F}_{uu} \cdot \hat{n} & \bar{F}_{uv} \cdot \hat{n} \\ \bar{F}_{uv} \cdot \hat{n} & \bar{F}_{vv} \cdot \hat{n} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix},$$

which is equivalent

$$\bar{f}'' \cdot \hat{n} = \bar{W}^T A \bar{W} ,$$

where B is the well known second fundamental matrix .

Definition 2.3.12. [13] The principal curvatures of S, denoted by k_1 and k_2 ,are the eigenvalues of the matrix $A^{-1}B$.

Definition 2.3.13.[13] The Gaussian curvature of S is $K = k_1 k_2$.

Definition 2.3.14. [13] The mean curvature of S is $M = \dfrac{k_1 k_2}{2}$.

Definition 2.3.15. [13] A point p on a surface S is called an umbilic point if $k_1(p) = k_2(p)$

Definition 2.3.16. [13] A surface S is *convex* if the Gaussian curvature K is strictly positive at every point of S.

Example 2.3.17. For the parameterization

$$\bar{F}(u,v) = (3\cos u \sin v, 3\sin u \sin v, 3\cos v),$$

$$A^{-1}B = \begin{bmatrix} \frac{1}{9\sin^2 v} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 3\sin^2 v & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

So the principal curvatures of S are $k_1 = \frac{1}{3}, k_2 = \frac{1}{3}$.

The Gaussian curvature is $K = k_1 k_2 = \frac{1}{9}$,

hence the surface S with parameterization $\bar{F}$ is convex .

The mean curvature of S is $M = \frac{k_1 + k_2}{2} = \frac{1}{3}$.

Every point on the sphere is umbilic.

Example 2.3.18. For the parameterization

$$\bar{F}(u,v) = (3\cos u, 3\sin u, v),$$

$$A^{-1}B = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & 0 \\ 0 & 0 \end{bmatrix}.$$

S0 the principal curvatures of S are $k_1 = 0, k_2 = \frac{-1}{3}$.

The Gaussian curvature is $K = k_1 k_2 = 0$,

hence the surface S with parameterization $\bar{F}$ is not convex .

The mean curvature of S is $M = \frac{k_1 + k_2}{2} = \frac{-1}{6}$.

There are no umbilic points

2.4 Surfaces of Constant Width in R^3

Suppose that S is a smooth compact connected surface in R^3 without boundary, so S is closed. If the Gaussian curvature K of S is every where positive, then S is diffeomorphic to the sphere S^2 and called an ovaloid. If T is a tangent plane of the ovaloid S at a point p, then there is exactly one other tangent plane T^* of S at some other p^* that is parallel to T . The perpendicular distance $a(p)$ between T and T^* is called the width of S at p (or p^*).The points p and p^* are said to be opposite to one another.

Definition 2.4.1. [12] An ovaloid S is said to be of constant width a if a is independent of p.

Theorem 2.4.2. [1] Let $\bar{f}: R \to R^2$ defined by $\bar{f}(u) = (f_1(u), f_2(u))$ be a curve of constant width a with $\bar{f}(u^*)$ as an opposite point of $\bar{f}(u)$. Assume that $\bar{f}$ is symmetric about a certain line in R^2 If $\bar{f}$ is rotated about that line of symmetry, then the surface of revolution $\bar{F}$ is of constant width a with $\bar{F}(u^*, v)$ being the opposite point of $\bar{F}(u \ \ , v)$ where v is the angle of rotation.

Note 2.4.3. Without loss of generality, the rotation by an angle v can be taken as rotation about the $x-axis, v \in [0, \pi]$. Hence

$$\bar{F}(u,v) = (\bar{f}_1(u), \bar{f}_2(u)\cos v, \bar{f}_2(u)\sin v).$$

Example 2.4.4. consider the unit circle

$$\bar{f}(u) = (\cos u, \sin u), u \in R$$

Of course $\bar{f}$ is a curve of constant width 2. The surface of revolution $\bar{F}$ is a sphere and defined by

$$\bar{F}(u,v) = (\cos u, \sin u \cos v, \sin u \sin v), v \in [0, \pi].$$

By Theorem **2.4.2** $\bar{F}(u,v)$ and $\bar{F}(u+\pi, v)$ are opposite points and the surface is of constant width 2.

Now

$$\bar{F}_u = (-\sin u, \cos u \cos v, \cos u \sin v)$$

and

$$\bar{F}_v = (0, -\sin u \sin v, \sin u \cos v),$$

and

$$\hat{n} = \frac{\bar{F}_u \times \bar{F}_v}{\left\|\bar{F}_u \times \bar{F}_v\right\|} = (\cos u, \sin u \cos v, \sin u \sin v).$$

The first and the second fundamental matrices of $\bar{F}$ respectively are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 u \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -\sin^2 u \end{bmatrix}.$$

The eigenvalues of $A^{-1}B$ are $k_1 = -1, k_2 = -1$ and so the Gaussian curvature is $K = 1$ (everywhere positive).

Chapter Three
Euclidean Transnormality

We start with a review of differentiable manifolds.

3.1 Differentiable Manifolds

Let M be a topological space with dimension m. Assume that M is covered by the family of open sets $\{U_\alpha : \alpha \in \Delta\}$, each open set U_α is equipped with a homeomorphism ϕ_α where $\phi_\alpha : U_\alpha \to R^m$, that is ϕ_α maps U_α onto some open set in R^m.

Definition 3.1.1. [7] An *m- manifold* M is a topological space with dimension m such that for all x in M, there exists an open set U_x in M that is homeomorphism to open set in R^m. Each U_x , together with a particular homeomorphism ϕ_x , is called a *patch.*

The collection $A = \{(U_\alpha, \phi_\alpha) : \alpha \in \Delta\}$ of all patches is called an *atlas* for M .

Example 3.1.2. The Euclidean space R^m itself is an m-manifold. Here we take $A = \{(R^m, Id)\}$ where Id is the identity map.

Example 3.1.3 Let M be an m-manifold and O any open set in M. Then O is a manifold. If $A = \{(U_\alpha, \phi_\alpha) : \alpha \in \Delta\}$ is an atlas for M, then

$B = \{(U_\alpha \cap O, \phi_\alpha| \ U_\alpha \cap O) : \alpha \in \Delta\}$ is an atlas for O.

Example 3.1.4. The circle $S^1 = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$ is a 1-manifold. Take the open sets U_1, U_2, U_3, U_4 to be the subsets of S^1 for which $x \succ 0, x \prec 0, y \succ 0, y \prec 0$ respectively. Define a homeomorphism ϕ_i form U_i

to the open interval $(-1,+1)$ in R to be

$$\phi_1(x, y) = \phi_2(x, y) = y, \ \phi_3(x, y) = \phi_4(x, y) = x .$$

Another atlas for S^1 can be built as follows. Take the two open set

$$U_+ = \{(x, y) \in S^1 - (0,-1)\} \quad \text{and} \quad U_- = \{(x, y) \in S^1 - (0,+1)\} .$$

Let $\phi_+(x, y) = \dfrac{x}{1+y}$ and $\phi_-(x, y) = \dfrac{x}{1-y}$.

The formulae for $\phi_\pm$ express the fact that $\phi_\pm$ are the stereographic projections from the circle S^1 minus the points $(0,\pm 1)$ onto the *x*-axis thought of as a copy of R. Thus, $\phi_\pm$ are homeomorphisms. Moreover, $A = \{(U_+, \phi_+), (U_-, \phi_-)\}$ gives an easy scene for the generalization in the next example.

Example 3.1.5. Consider the *m*-sphere $S^m = \{(x_1, ..., x_{m+1}) \in R^{m+1} : \sum_{i=1}^{m+1} x_i^2 = 1\}$.

Take $U_+ = \{(x_1,...,x_{m+1}) \in S^m - (\underline{0},-1)\}$

and $U_- = \{(x_1,...,x_{m+1}) \in S^m - (\underline{0},+1)\}$.

Let $$\phi_+(x_1,...,x_{m+1}) = \frac{1}{1+x_{m+1}}(x_1,...,x_m)$$

and $$\phi_-(x_1,...,x_{m+1}) = \frac{1}{1-x_{m+1}}(x_1,...,x_m).$$

Then the stereographic projections $\phi_+ : U_+ \to R^m$ and $\phi_- : U_- \to R^m$ are homeomorphisms.

Definition 3.1.6. [7] An atlas $\mathrm{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in \Delta\}$ of a manifold M is a C^r atlas if for every α, β, the map $\phi_\alpha \phi_\beta^{-1}$ is of class C^r. A maximal C^r atlas is a C^r atlas whose patches overlap C^r differentiably with its own patches. In this case A is a C^r *structure* for M. Also M it self is a C^r differentiable manifold.

Definition 3.1.7 [7] Let M and N be C^r differentiable manifolds. Then a C^r diffeomorphism f from M to N is a homeomorphism $f : M \to N$ which is a C^r map and whose inverse $f : N \to M$ is a C^r map.

Remark 3.1.8. If M and N are manifolds of dimensions m,n respectively, then their Cartesian product $M \times N$ is an $(m+n)$-manifold generating a smooth structure from the smooth structures on M and N. Some examples are the torus $S^1 \times S^1$ and the cylinder $S^1 \times R$.

Definition 3.1.9. [7] Let M be a smooth m-manifold in R^n. Then a subset K of M is a submanifold of M of dimension k if every point of K lies in some patch (U,ϕ) of M with ϕ taking $K \cap U$ to $R^k \cap \phi(U)$.

Definition 3.1.10. [7] Let m,n be positive natural numbers, U be an open subset of R^n and $F : U \to R^m$ be a C^r–map. A point $p \in U$ is said to be *critical* for F if the determinant of $dF_p \cdot (dF_p)^t \neq 0$ where dF_p is the differential of F ,and *regular* if it is not critical. A point $q \in F(U)$ is said to be *a regular value* of F if every point of the pre-image $F^{-1}(\{q\})$ of q is regular and *a critical value* otherwise.

Theorem 3.1.11.[7] (***The Implicit Function Theorem***). Let m,n be natural numbers such that $m \prec n$ and $F : U \to R^m$ be a C^r–map from an open subset U of R^n. If $q \in F(U)$ is a regular value of F then the pre-image $F^{-1}(\{q\})$ of q is an $(n-m)$-dimensional submanifold of R^n of class C^r.

Employing the implicit function theorem, we get the following interesting example of the m-dimensional sphere S^m as a differentiable submanifold of R^{m+1}.

Example 3.1.12. Let $F: R^{m+1} \to R$ be the C^∞-map given by

$$F:(p_1,\ldots,p_{m+1}) \to \sum_{i=1}^{m+1} p_i^2 \ .$$

The differential dF_p of F at p is given by $dF_p = 2p$, so

$$dF_p \cdot (dF_p)^t = 4|p|^2 \in R \, .$$

This means that $1 \in R$ is a regular value of F so the fibre

$$S^m = \{p \in R^{m+1} \mid |p|^2 = 1\} = F^{-1}(\{1\})$$

of F is an m-dimensional submanifold of R^{m+1}. It is the standard m-dimensional sphere introduced in Example 3.1.5.

3.2 Transnormality

The idea of transnormality is a generalization of the concept of an *m*-hypersurface of constant width in R^{m+1}, it is due to S. Robertson [14,15,16] and contributions have been made by S.Carter and K.Al-Banawi [1,2,3,5,8].

The notion of constant width can be formulated as follows. Let *M* be a smooth compact connected *m*-manifold without boundary that is smoothly embedded in R^{m+1}. A chord of *M* is normal if it is normal to *M* at one of its endpoints and binormal if it is normal to *M* at both end points.

The manifold *M* is of constant width if and only if every normal chord of *M* is binormal to *M*. Each point of the endpoints is called the *opposite* of the other.

Let *M* be a smooth connected *m*-manifold without boundary and let $f: M \to R^n$ be a smooth embedding of *M* into R^n. Let *V*= *f(M)*. For each point $p \in V$ there exists a unique tangent plane T_pV tangent to *V* at *p* with dimension *m* and a unique normal plane N_pV normal to *V* at *p* with dimension *n-m*. Thus, there are maps *T* and *N* with *T*(*p*) = T_pV and
N(*p*) = N_pV.

Definition 3.2.1. [14] The *m*-manifold *V* is *transnormal* in R^n iff $\forall p, q \in V$, if $q \in$ *N(p)*, then *N(q)* = *N(p)*.

Let *W* be the space of normal planes of *V*, say *W* = *N(V)*. S.Robertson showed that for any transnormal embedding *V* in R^n, the order of *N* as a covering map is always finite [16]. If *V* is transnormal in R^n and the order of *N* is *r*, then *V* is called an *r*-transnormal manifold.

Definition 3.2.2. [14] Let *V* be a transnormal manifold in R^n. The *generating frame* of *V* at *p* is

$$\phi(p) = V \cap N(p) \ .$$

If *V* is *r*-transnormal, then $|\ \phi(p)\ | = r$ where | … | is the cardinality.

It is true that any two generating frames are isometric .That is , if $\phi(p_1)$ and $\phi(p_2)$ are generating frames, then there exists $F:\phi(p_1)\rightarrow\phi(p_2)$ which preserves distance. Also if *V* is a compact *r*-transnormal manifold, then *r* is even [15].

3.3 The Distance Function on a Transnormal Manifold

Let M be a smooth connected m-manifold without boundary. Assume that is a smooth embedding of M into R^n. Let *V*= *f*(M). Assume that V is r-transnormal in R^n. For $p\in V$, let $\Lambda_p : V\rightarrow R$ be the distance function measuring the distance from p. Then Λ_p has a critical point at *q* iff q is in *N(p)* [16]. Thus, the set of critical points of Λ_p is the generating frame $\phi(p)$.

Example 3.3.1. Consider the embedding f of S^2 in R^3 defined by

$$f(\theta,\phi)=(\cos\theta\cos\phi,\sin\theta\cos\phi,\sin\phi)$$

with a suitable domain. The embedding is 2-transnormal in R^3. The generating frame at $f(\theta_0,\phi_0)$ is { $f(\theta_0,\phi_0)$, $f(\theta_0+\pi,-\phi_0)$ }.

Let $L(\theta,\phi)=\Lambda_{f(\theta_0,\phi_0)}(f(\theta,\phi))=2-2\cos\phi\cos\phi_0\cos(\theta-\theta_0)-2\sin\phi\sin\phi_0$.

Now

$$L_\theta=2\cos\phi\cos\phi_0\sin(\theta-\theta_0)$$

and

$$L_\phi=2\sin\phi\cos\phi_0\cos(\theta-\theta_0)-2\cos\phi\sin\phi_0.$$

By equating L_θ and L_ϕ to zero, one gets

$$\theta=\theta_0\Rightarrow\phi=\phi_0 \text{ and } \theta=\theta_0+\pi\Rightarrow\phi=-\phi_0 .$$

Thus, the set of critical points of $L(\theta,\phi)$ is the generating frame at $f(\theta_0,\phi_0)$.

Theorem 3.3.2. [16] The sphere S^2, the torus T^2 and the Klein bottle K^2 are the only transnormal surfaces that can exist .

Theorem 3.3.3. [16] The sphere S^2 is the only transnormal surface that can exist in R^3 . Also S^2 is always 2 -transnormal in R^3.

Theorem 3.3.4. [16] A 2-transnormal compact surface in R^4 is a sphere.

Theorem 3.3.5. [5] A 4-transnormal compact surface in R^4 is a torus.

Theorem 3.3.6. [5] Let V be an *r*-transnormal torus in R^4. If *W*, the space of normal planes to *V*, is *orientable*, then $r=4$ and the generating frame of V is the set of vertices of a rectangle, *possibly a square*.

Theorem 3.3.7. [16] Let V_1 be an r_1-transnormal manifold of dimension m_1 in R^{n_1} and V_2 an r_2-transnormal manifold of dimension m_2 in R^{n_2}. Then the product $V_1\times V_2$ is an r_1r_2-transnormal manifold of dimension m_1+m_2 in $R^{n_1+n_2}$ where $R^{n_1+n_2}$ is identified with $R^{n_1}\times R^{n_2}$.

Chapter Four
Transnormal Spherical Partial Tubes

4.1 Introduction

The general definition of a *partial tube* was introduced in [9] as follows. Let M be a smooth connected m-manifold without boundary. Let $f : M \rightarrow R^n$ be a smooth embedding of M into the Euclidean space R^n, $n = m + k$. For $p \in M$, let T_pM be the tangent plane of M at p. Consider the normal bundle of M

$$\aleph = \{(p,v) : p \in M, v \perp T_pM\}$$

and the smooth endpoint map $\eta : \aleph \rightarrow R^n$ defined by

$$\eta(p,v) = p + v.$$

Let $\sum$ be the set of singular points of η. Let $P \subset \aleph$ be a smooth subbundle with type fibre S such that S is a smooth submanifold of R^k. If $P \cap \Sigma$ is empty, then P is a smooth manifold and $\eta \mid P$ is a smooth embedding called a *partial tube* around f. The manifold $V=f(M)$ is usually called the *base* of the partial tube h. A partial tube is *spherical* if S is a sphere. The word partial is used if S is embedded in a proper subplane of the normal plane at p. Otherwise, the spherical tube is called a *full tube*. Embeddings similar to h with S being an image of an embedding were studied in [9].

4.2 Radius of Transnormal Spherical Partial Tubes

Assume that $f : M \rightarrow R^n$ is a smooth r-transnormal embedding of the compact connected m-manifold M without boundary in the Euclidean space R^n. Then the next theorem ensures the existence of $\xi > 0$ such that the above full tube is the image of an embedding. Also if $p \in V = f(M)$, then the normal plane of V at p, $N(p)$, intersects the full tube at points based at the points of the generating frame $\phi(p)$. By a similar argument this result holds if the normal bundle $\aleph$ is replaced by a subbundle P of $\aleph$.

Theorem 4.2.1. [1] Let $f : M \rightarrow R^n$ be a smooth r-transnormal embedding of the compact connected m-manifold M without boundary into the Euclidean space R^n. Then for some $\xi > 0$ sufficiently small

(1). The map $\eta | \aleph^{\xi} V$ is an embedding, and

(2). For all $p \in V$, for all $(q,v) \in \eta | \aleph^{\xi} V$,

$$\eta(q,v) \in N(p) \text{ iff } q \in N(p) \cap V.$$

In the next theorem the dimension of the normal plane is the sum of the dimension of the parallel normal plane (d) and the dimension of its compliment (k).

Theorem 4.2.2. [3] Let f : $M\to$ R^{m+d+k} be a smooth r-transnormal embedding of the compact connected m-manifold M without boundary into the Euclidean space R^{m+d+k}. Then there exists a $2r$-transnormal embedding of a $(k-1)$-sphere bundle over $V=f(M)$ in R^{m+d+k} with image a partial tube and V is its base.

Definition 4.2.3. Let h be a spherical partial tube about f. Then the radius of h is the radius of the $(k-1)$-sphere S over f.

Now we explain how to get a suitable condition regarding the radius of h.

Let $(x_1,\ldots,x_m)$ be local coordinates of M and $p\in M$. Also let $\{v_1(p),\ldots,v_d(p),v_{d+1}(p),\ldots,v_{d+k}(p)\}$ be the set of orthonormal vectors that span the normal plane of $V=f(M)$ at p where $v_1(p),\ldots,v_d(p)$ are parallel . Then the connection equations of the orthonormal field of f are

$$\frac{\partial v_1}{\partial x_i}=\sum_{u=1}^{m}a_{liu}\frac{\partial f}{\partial x_u}+\sum_{u=d+1}^{d+k}b_{liu}v_u, i=1,\ldots,m, l=1,\ldots,d+k$$

where a_{liu}, b_{liu} denote the connection forms.

Let $(y_1,\ldots,y_{k-1})$ be local co-ordinates of $U\subset R^{k-1}$ and $q\in U$. Also let $g:U\to R^k$ be a smooth 2 – transnormal embedding of U in the Euclidean space R^k such that $g(U)$ is a $(k-1)$-sphere of radius ξ. Now consider the partial tube in R^{m+k+d} defined by

$$h(p,q)=f(p)+\sum_{l=d+1}^{d+k}g_l(q)v_l(p).$$

The evaluation will be at the point (p,q) which is omitted for simplicity.

Now for $i=1,\ldots,m$,

$$\begin{aligned}\frac{\partial h}{\partial x_i}&=\frac{\partial f}{\partial x_i}+\sum_{l=d+1}^{d+k}g_l\sum_{u=1}^{m}a_{liu}\frac{\partial f}{\partial x_u}+\sum_{l=d+1}^{d+k}g_l\sum_{u=d+1}^{d+k}b_{liu}v_u\\&=\sum_{u=1}^{m}\beta_{iu}\frac{\partial f}{\partial x_u}+\sum_{u=d+1}^{d+k}\alpha_{iu}v_u\end{aligned}$$

where

$$\beta_{ij}=\begin{cases}\sum_{l=d+1}^{d+k}g_la_{lij} & i\neq j\\ 1+\sum_{l=d+1}^{d+k}g_la_{lij} & i=j\end{cases}$$

and

$$\alpha_{ij}=\sum_{l=d+1}^{d+k}g_lb_{lij}.$$

Put $C_l = (a_{lij}), B = (\beta_{ij}), G = (\frac{\partial g_i}{\partial y_j})$. Then

$$B = I_m + \sum_{l=d+1}^{d+k} g_l C_l.$$

Let $\Gamma(f)$ be the first fundamental form of f, and let $\Pi_{v_l}(f)$ be the second fundamental form of f in the direction of v_l. Then for $l = d+1,\dots,d+k, j = 1,\dots,m,$

$$\Pi_{v_l}(f) = (\langle \frac{\partial^2 f}{\partial x_i \partial x_j}, v_l \rangle) = -(\langle \frac{\partial f}{\partial x_i}, \frac{\partial v_l}{\partial x_j} \rangle) = -(\langle \frac{\partial f}{\partial x_i}, \sum_{u=1}^{m} a_{lju} \frac{\partial f}{\partial x_u} \rangle)$$

$$= -(\sum_{u=1}^{m} a_{lju} \langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_u} \rangle) = -[C_l \Gamma(f)]_{ji} = -[\Gamma(f) C_l^T]_{ij}$$

where C_l^T is the transpose matrix of C_l.
Hence

$$C_l^T = -\Gamma^{-1}(f)\Pi_{v_l}(f),$$

i.e.

$$C_l = -A_{v_l}^T(f)$$

where A_{v_l} is the shape operator. Thus,

$$B = I_m - \sum_{l=d+1}^{d+k} g_l A_{v_l}^T(f) = (I_m - \sum_{l=d+1}^{d+k} g_l A_{v_l}(f))^T .$$

Now for $i, j = 1,\dots,m,$

$$\langle \frac{\partial h}{\partial x_i}, \frac{\partial h}{\partial x_j} \rangle = \langle \sum_{u=1}^{m} \beta_{iu} \frac{\partial f}{\partial x_u} + \sum_{u=d+1}^{d+k} \alpha_{iu} v_u, \sum_{u=1}^{m} \beta_{ju} \frac{\partial f}{\partial x_u} + \sum_{u=d+1}^{d+k} \alpha_{ju} v_u \rangle$$

$$= \langle \sum_{u=1}^{m} \beta_{iu} \frac{\partial f}{\partial x_u}, \sum_{u=1}^{m} \beta_{ju} \frac{\partial f}{\partial x_u} \rangle + \langle \sum_{u=d+1}^{d+k} \alpha_{iu} v_u, \sum_{u=d+1}^{d+k} \alpha_{ju} v_u \rangle$$

$$= \sum_{u=1}^{m} \beta_{iu} \sum_{t=1}^{m} \beta_{jt} \langle \frac{\partial f}{\partial x_u}, \frac{\partial f}{\partial x_t} \rangle + \sum_{u=d+1}^{d+k} \alpha_{iu} \alpha_{ju}$$

$$= \sum_{u=1}^{m} \beta_{iu} [B\Gamma(f)]_{ju} + [DD^T]_{ij}$$

$$= \sum_{u=1}^{m} \beta_{iu} [\Gamma(f) B^T]_{uj} + [DD^T]_{ij}$$

$$= [B\Gamma(f)B^T]_{ij} + [DD^T]_{ij}$$

$$= [(I_m - \sum_{l=d+1}^{d+k} g_l A_{v_l}(f))^T \Gamma(f)(I_m - \sum g_l A_{v_l}(f)) + [DD^T]_{ij}.$$

Also for $i, j = 1,\ldots,k-1$,

$$\langle \frac{\partial h}{\partial y_i}, \frac{\partial h}{\partial y_j} \rangle = \langle \sum_{l=d+1}^{d+k} \frac{\partial g_l}{\partial y_i} v_l, \sum_{l=d+1}^{d+k} \frac{\partial g_l}{\partial y_j} v_l \rangle$$
$$= \sum_{l=d+1}^{d+k} \sum_{\lambda=d+1}^{d+k} \frac{\partial g_l}{\partial y_i} \frac{\partial g_\lambda}{\partial y_\lambda} \langle v_l, v_\lambda \rangle$$
$$= \sum_{l=d+1}^{d+k} \frac{\partial g_l}{\partial y_i} \frac{\partial g_l}{\partial y_j} = [G^T G]_{ij}.$$

Also for $i = 1,\ldots,m,\ j = 1,\ldots,k-1$,

$$\langle \frac{\partial h}{\partial x_i}, \frac{\partial h}{\partial y_j} \rangle = \langle \sum_{u=1}^{m} \beta_{iu} \frac{\partial f}{\partial x_u} + \sum_{u=d+1}^{d+k} \alpha_{iu} v_u, \sum_{l=d+1}^{d+k} \frac{\partial g_l}{\partial y_j} v_l \rangle$$
$$= \langle \sum_{u=d+1}^{d+k} \alpha_{iu} v_u, \sum_{l=d+1}^{d+k} \frac{\partial g_l}{\partial y_j} v_l \rangle$$
$$= \sum_{u=d+1}^{d+k} \sum_{l=d+1}^{d+k} \alpha_{iu} \frac{\partial g_l}{\partial y_j} \langle v_u, v_l \rangle$$
$$= \sum_{u=d+1}^{d+k} \alpha_{iu} \frac{\partial g_u}{\partial y_j} = [DG]_{ij},$$

Thus, the matrix $\Gamma(h)$ is

$$\begin{pmatrix} (I_m - \sum_{l=d+1}^{d+k} g_l A_{v_l}(f))^T \Gamma(f)(I_m - \sum_{l=d+1}^{d+k} g_l A_{v_l}(f)) + DD^T & DG \\ (DG)^T & G^T D \end{pmatrix}.$$

Theorem 4.2.4 Let $f: M \to R^{m+d+k}$ be an r-transnormal embedding of a compact, connected m-manifold M without boundary. Then the spherical partial tube h about f is a $2r$-transnormal embedding iff $\Gamma(h)$ is non-singular.

Proof: Since the map h is one to one, its domain is compact and its image is in a Hausdorff space, the partial tube h is an embedding iff h is an immersion iff $\Gamma(h)$ is non-singular.

4.3 An Example

We start with a curve of constant width *2a* in R^4. Then we construct a 4-transnormal embedding of a torus around such a 2-transnormal curve in R^4. We explain how to construct the torus and show that its generating frame is the set of vertices of a rectangle, one of its lengths is *2a* and the other is the radius in Definition 2.4.3.

Assume that the curve is defined by $f:[0,2\pi]\to R^4$ with a parameter t. Take $f(t_1+\pi)$ as the opposite point of $f(t_1)$ where $t_1\in[0,2\pi]$ and $t_1+\pi$ is taken mod 2π. Let v_0, v_1, v_2 and v_3 be an orthonormal field along f where $v_0(t_1)$ is the unit tangent of f at $f(t_1)$, i.e. $v_0(t_1)=\dfrac{f'(t_1)}{\|f'(t_1)\|}$, and the vectors $v_1(t_1)$, $v_2(t_1)$ and $v_3(t_1)$ span the normal plane of f at $f(t_1)$, denoted by $N_f(t_1)$. Let $v_1(t_1)=\dfrac{f(t_1+\pi)-f(t_1)}{2a}$. Then

$v_1'(t_1)=-\dfrac{\|f'(t_1)\|}{a}v_0(t_1)$, i.e. v_1 is parallel.

Let ξ be chosen as in Theorem 4.2.1. Then there exists $n\in N$ such that $\dfrac{a}{n}<\xi$. Take $b=\dfrac{a}{n}$. Now let $g:[0,2\pi]\to R^2$ be the circle defined by $g(s)=(b\cos s, b\sin s)$. Clearly, $g(s+\pi)$ is the opposite point of the point $g(s)$ on the circle where s is taken mod 2π. For each $t_1\in[0,2\pi]$, put the above circle in R^4 such that it is centred at $f(t_1)$ in the plane spanned by $v_2(t_1)$ and $v_3(t_1)$. The circles fit together to form a partial tube defined by

$$h(t,s)=f(t)+bv_2(t)\cos s+bv_3(t)\sin s \tag{1}$$

By Theorem 4.2.1, h is an embedding.

The tangent plane to h at $h(t_1,s_1)$, denoted by $T_h(t_1,s_1)$, is spanned by

$$h_t(t_1,s_1)=f'(t_1)+bv_2'(t_1)\cos s_1+bv_3'(t_1)\sin s_1,$$
$$h_s(t_1,s_1)=-bv_2(t_1)\sin s_1+bv_3(t_1)\cos s_1.$$

The normal plane to h at $h(t_1,s_1)$, denoted by $N_h(t_1,s_1)$, is spanned by

$$\bar{v}_1(t_1,s_1)=v_1(t_1) \tag{2}$$
$$\bar{v}_2(t_1,s_1)=-(v_2(t_1)\cos s_1+v_3(t_1)\sin s_1) \tag{3}$$

It worth to mention here that $\bar{v}_1(t_1,s_1)$ is independent of s_1, and so $\bar{v}_1(t_1,s_1)$ is perpendicular to the circle centred at $f(t_1)$ at every point. Let Im*(h)* be the image of h and Im*(g_t)* be the circle in the partial tube centred at $f(t)$. Let $\Delta=N_h(t_1,s_1)\cap \mathrm{Im(h)}$. Let $s_2\in[0,2\pi]$ such that $\bar{v}_2(t_1+\pi,s_2)=\bar{v}_2(t_1,s_1)$. Then

$$\Delta=\{h(t_1,s_1),h(t_1,s_1+\pi),h(t_1+\pi,s_2),h(t_1+\pi,s_2+\pi)\} \tag{4}$$

Now the partial tube h in (1) is a $4-$transnormal torus and its generating frame is the set (4), which is a set of vertices of a rectangle with lengths $2a$ (the width of f) $2b$ (the radius of the torus).

The next is a numerical example.

Example 4.2.5. Consider the embedding f in R^4 defined by

$$f(t) = (6\cos t, 6\sin t, 0, 0)$$

where t is taken mod 2π. The image of f is a circle of radius 6, and so f is 2-transnormal in R^4 with constant width 12 and $f(t+\pi)$ is the opposite point of $f(t)$. An orthonormal field along f is

$$v_0(t) = (-\sin t, \cos t, 0, 0),$$
$$v_1(t) = (-\cos t, -\sin t, 0, 0),$$
$$v_2(t) = (0, 0, 1, 0),$$
$$v_3(t) = (0, 0, 0, 1).$$

Take $b = 3$. Then the partial tube

$$h(t,s) = (6\cos t, 6\sin t, 3\cos s, 3\sin s)$$

is a 4-transnormal embedding of a torus in R^4 for which f is the base. The generating frame of the torus at $h(t_1, s_1)$ is the set

$$\Delta = \{h(t_1, s_1), h(t_1, s_1 + \pi), h(t_1 + \pi, s_1), h(t_1 + \pi, s_1 + \pi)\}.$$

Moreover, the radius of h is 6.

REFERENCES

[1] Al-Banawi. K. (2004) **Generating Frames and Normal Holonomy of Transnormal Submanifolds in Euclidean Space***s*. Unpublished PhD Thesis, University of Leeds.

[2] Al-Banawi and Carter. S. (2004), Generating Frames of Transnormal Curves. **Soochow Journal of Mathematics**, **30** (3) 261-268.

[3] Al-Banawi K. and Carter. S. (2005), Transnormal Partial Tubes. **Contributions to Algebra and Geometry**, **46** (2) 575-580.

[4] Al-Banawi. K. (2008), Geometry of Ovals in R^2 in Terms of the Support Function, **Journal of Applied Sciences**, **8**(2):383-386.

[5] Al-Banawi. K. (2009), Focal Point of 4- Transnormal Tori in R^4, **Georgian Mathematical Journal***,* **16**(2) 211-218.

[6] Al-Banawi K. and Al-Btoush R. (2011), Some Geometric and Analytic Properties of Closed Convex Curves in *R2*, **Dirasat, Pure Sciences***,* **38**(1):81-84.

[7] Berger M. and Gostiaux. B. (1988) **Differential Geometry**: **Manifold, Curves and Surfaces***,* translated by Silvio Levy. Springer-Verlag, New York.

[8] Carter. S. (1969), A Class of Compressible Embeddings. *Proc.* **Camb. Phil. Soc**., **65** 23-26.

[9] Carter S. and West. A. (1995), Partial Tubes about Immersed Manifolds.**Geometriae Dedicata**, **54** 145-169.

[10] Eggleston. H. (1958), **Convexity**, Cambridge Tracts in Mathematics, 47, Cambridge University Press.

[11] Fillmore. J. (1969), Symmetries of Surfaces of Constant Width, **J. Differential Geometry**, **3** : 103-110.

[12] Mellish. A. (1931), Notes on Differential Geometry**, Ann. Math**., **32** : 181-190.

[13] O'Neill. B. (1997), **Elementary Differential Geometry** (2nd Edition), Academic Press, London.

[14] Robertson. S. (1964), Generalized Constant width for Manifolds. **Michigan Math. J., 11** 97-105.

[15] Robertson. S. (1967), On Transnormal Manifolds. **Topology**, **6** 117-123.

[16] Robertson. S. (1984), Smooth Curves of Constant Width and Transnormality. **Bull. London Math. Soc., 16** 264-274.